A Strong Invariance Theorem of the Tail Empirical Copula Processes

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Abstract

We study the behavior of bivariate empirical copula process $\mathbb{G}_n(\cdot,\cdot)$ on pavements $[0,k_n/n]^2$ of $[0,1]^2$, where k_n is a sequence of positive constants fulfilling some conditions. We provide a upper bound for the strong approximation of $\mathbb{G}_n(\cdot,\cdot)$ by a Gaussian process when $k_n/n \searrow \gamma$ as $n \to \infty$, where $0 \leq \gamma \leq 1$.

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1 Introduction

Let $\{(X_n,Y_n) : n \geq 1\}$ be independent replicæ of a random vector (X,Y) with distribution function [df] $\mathbb{F}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$. We assume that the corresponding marginal df's $G(x) = \mathbb{P}(X \leq x)$ and $H(y) = \mathbb{P}(Y \leq y)$ are continuous. It is well know that there exists a distribution function $\mathbb{C}(\cdot,\cdot)$ with uniform marginals on $[0,1]^2$ such that

$$\mathbb{F}(x,y) = \mathbb{C}(G(x),H(y)) \quad \text{for} \quad (x,y) \in \mathbb{R}^2.$$

See Sklar (1959, 1973), Moore and Spruill (1975), Deheuvels (1979). The function $\mathbb{C}(\cdot,\cdot)$ is called the copula associated with $\mathbb{F}(\cdot,\cdot)$ (some authors called it the dependence function). This function fulfills the

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following identity

$$\mathbb{C}(u,v) = \mathbb{F}(G^{-1}(u), H^{-1}(v)) \quad \text{for} \quad (u,v) \in [0,1]^2, \tag{1.1}$$

where $G^{-1}(u) = \inf\{x : G(x) \geq u\}$ and $H^{-1}(v) = \inf\{y : H(y) \geq v\}$ are the quantile functions pertaining, respectively, to $G(\cdot)$ and $H(\cdot)$. In the monographs by Nelsen (2006) and Joe (1997) the reader may find detailed ingredients of the modelling theory as well as surveys of the commonly used copulas. The empirical counterparts of $\mathbb{F}(\cdot,\cdot), G(\cdot)$ and $H(\cdot)$, based upon $(X_1,Y_1),\ldots,(X_n,Y_n)$, are given, respectively, for each $n \geq 1$ and $x, y \in \mathbb{R}$, by

$$\mathbb{F}_n(x,y) := \frac{1}{n}\sum_{i=1}^{n} 1\!\mathrm{I}\{X_i \leq x, Y_i \leq y\},$$

$$G_n(x) := \mathbb{F}_n(x,\infty) = \frac{1}{n}\sum_{i=1}^{n} 1\!\mathrm{I}\{X_i \leq x\} \;\text{ and }\; H_n(y) := \mathbb{F}_n(\infty,y) = \frac{1}{n}\sum_{i=1}^{n} 1\!\mathrm{I}\{Y_i \leq y\},$$

where $1\!\mathrm{I}\{\cdots\}$ denotes the indicator function of the set $\{\cdots\}$. In view of the characterization (1.1), we define an empirical copula function of $\mathbb{F}_n(\cdot,\cdot)$ by

$$\mathbb{C}_n(u,v) := \mathbb{F}_n(G_n^{-1}(u), H_n^{-1}(v)), \tag{1.2}$$

where $G_n^{-1}(u) := \inf\{x : G_n(x) \geq u\}$ and $H_n^{-1}(v) := \inf\{y : H_n(y) \geq v\}$, for $0 \leq u, v \leq 1$ and $n \geq 1$, denote the empirical quantile functions of $G_n(\cdot)$ and $H_n(\cdot)$, respectively.
We may now define the empirical copula process $\mathbb{G}_n(\cdot,\cdot)$ by setting

$$\mathbb{G}_n(u,v) := n^{1/2}(\mathbb{C}_n(u,v) - \mathbb{C}(u,v)), \quad \text{for} \quad (u,v) \in [0,1]^2. \tag{1.3}$$

The asymptotic behavior of the copula process $\mathbb{G}_n(\cdot,\cdot)$ has been investigated in various setting by many authors in the context of process convergence. Deheuvels (1979) investigated the consistency of $\mathbb{C}_n(\cdot,\cdot)$ and Deheuvels (1980, 1981) proved a functional central limit theorem for $\mathbb{G}_n(\cdot,\cdot)$ in the particular case of independent margins. Rüschendorf (1974, 1976), Gaenssler and Stute (1987) proved weak convergence of the empirical copula process $\mathbb{G}_n(\cdot,\cdot)$ in the Skorokhod space $D([0,1]^2)$. Van der Vaart and Wellner (1996) established weak convergence in the space $\ell^\infty([a,b]^2)$, when $0 < a < b < 1$. Fermanian *et al.* (2004) showed that the weak convergence of $\mathbb{G}_n(\cdot,\cdot)$ to a centered Gaussian process $\mathbb{G}(\cdot,\cdot)$ holds on $\ell^\infty([0,1]^2)$, when $\mathbb{C}(\cdot,\cdot)$ has continuous partial derivatives on $[0,1]^2$. In particular, these conditions are satisfied under the independence assumption of margins, i.e., for $(u,v) \in [0,1]^2$

$$\mathbb{C}(u,v) = uv.$$

A natural question arises then: which is the rate for the strong approximation, V_n say, of empirical copula processes $\mathbb{G}_n(\cdot,\cdot)$ by sequences of Gaussian processes $\mathbb{B}_n(\cdot,\cdot)$ such that

$$\sup_{(u,v)\in[0,1]^2} |\, \mathbb{G}_n(u,v) - \mathbb{B}_n(u,v) \,| = \mathcal{O}(V_n), \quad \text{a. s.?}$$

This is known in the scientific literature under the name of *invariance principle*.
Under the independence assumption of margins, Deheuvels *et al.* (2006) proved that the strong invariance principle holds with $V_n = n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}$, where $\mathbb{B}_n(\cdot,\cdot)$ is a sequence of bivariate

tied-down Brownian bridges (see (2.2) below). In the multivariate case, i.e., the dimension $d \geq 3$, Deheuvels (2009) showed that the rate for the strong approximations of empirical copula process is $V_n = n^{-1/(2d)}(\log n)^{2/d}$ where $\mathbb{B}_n(\cdot)$ is a sequence of d-variate *tied-down Brownian bridges* (see, Theorem 2.1, p.140 of Deheuvels (2009)).
Set $\log_1 u = \log_1 u = \log(u \vee e)$, and $\log_p u = \log_1(\log_{p-1} u)$ for $p \geq 2$. Throughout this paper, $\{k_n\}_{n=1}^{\infty}$ will denote a sequence of positive constants such that for all integers $n > 1$,

(H.1) $0 < k_n \leq n$;

(H.2) $k_n \uparrow$ as $n \to \infty$;

(H.3) $k_n/n \searrow \gamma$ with $0 \leq \gamma \leq 1$ as $n \to \infty$;

(H.4) $k_n/\log_2 n \to \infty$, as $n \to \infty$.

In the present paper, we are mainly concerned with the empirical copula process defined, in terms of a sequence $\{k_n\}_{n=1}^{\infty}$, for each $n \geq 1$, by

$$\mathbb{G}_n^*(u,v) := \mathbb{G}_n\left(u\frac{k_n}{n}, v\frac{k_n}{n}\right) \quad \text{for } (u,v) \in [0,1]^2. \tag{1.4}$$

The remainder of the paper is organized as follows. In §2, we state the main result concerning the rate of uniform almost sure convergence of the process $\{\mathbb{G}_n^*(u,v) : (u,v) \in [0,1]^2\}$, $n \geq 1$, defined in (1.4), in the case of independent and general marginals. These results will be used, in §2.4, to derive some asymptotic properties of the statistic considered in this paper for testing tail independence. We also indicate the basic technical tools needed for establishing this result. In §3, we apply our result to the smoothed local empirical copula process, where we limit ourselves to the case of independent marginals.

2 Behavior of empirical copula process on pavements

2.1 Gaussian Process

Our main aim is to provide a strong approximation of the process $\mathbb{G}_n^*(\cdot,\cdot)$ based upon

$$(X_1, Y_1), \ldots, (X_n, Y_n)$$

by a sequence of Gaussian processes. To award this goal, we first need to introduce several approximating Gaussian processes, we refer for example to Deheuvels (2007).
Let $\{\mathbf{W}(s,t) : s \geq 0,\ t \geq 0\}$ be a two-time parameter Wiener process (or *Brownian sheet*), namely, a centered Gaussian process, with continuous trajectories and covariance function given by

$$\mathbb{E}(\mathbf{W}(s_1,t_1)\mathbf{W}(s_2,t_2)) = (s_1 \wedge s_2)(t_1 \wedge t_2), \quad \text{for } s_1, s_2, t_1, t_2 \geq 0,$$

where $(s_1 \wedge s_2) = \min(s_1, s_2)$.
A bivariate *Brownian bridge* is defined, in terms of the two-time Wiener process $\mathbf{W}(\cdot, \cdot)$, via

$$\mathbf{B}(s,t) = \mathbf{W}(s,t) - st\mathbf{W}(1,1), \quad \text{for } (s,t) \in [0,1]^2. \tag{2.1}$$

This process has continuous sample paths and fulfills

$$\mathbb{E}(\mathbf{B}(s,t)) = 0, \;\; \mathbb{E}(\mathbf{B}(s_1,t_1)\mathbf{B}(s_2,t_2)) = (s_1 \wedge s_2)(t_1 \wedge t_2) - \prod_{i=1}^{2} s_i t_i, \quad \text{for } (s_1, s_2, t_1, t_2) \in [0,1]^4.$$

A bivariate *tied-down Brownian bridge* is defined, in terms of the bivariate Brownian bridge $\mathbf{B}(\cdot, \cdot)$, via

$$\mathbb{B}(s,t) = \mathbf{B}(s,t) - s\mathbf{B}(1,t) - t\mathbf{B}(s,1), \quad \text{for } (s,t) \in [0,1]^2. \tag{2.2}$$

It is a centered process with continuous sample paths and covariance function given by

$$\mathbb{E}(\mathbb{B}(s_1,t_1)\mathbb{B}(s_2,t_2)) = \{s_1 \wedge s_2 - s_1 s_2\}\{t_1 \wedge t_2 - t_1 t_2\}, \quad \text{for } (s_1, s_2, t_1, t_2) \in [0,1]^4.$$

2.2 Asymptotic Theory

Throughout the sequel, we assume that X and Y are mutually independent with continuous distribution functions $G(\cdot)$ and $H(\cdot)$. Thus, $\{U_i = G(X_i)\}_{1 \leq i \leq n}$ and $\{V_i = H(Y_i)\}_{1 \leq i \leq n}$ are two independent sequences of independent and identically distributed uniform $(0,1)$ random variables. For all $(u,v) \in [0,1]^2$, the distribution function $\mathbb{T}(\cdot, \cdot)$ associated with (U,V) fulfills the following identity

$$\mathbb{T}(u,v) := \mathbb{P}(U \leq u, V \leq v) = \mathbb{F}(G^{-1}(u), H^{-1}(v)) = \mathbb{C}(u,v) = uv.$$

We define, for each $n \geq 1$ and $0 \leq u, v \leq 1$, the empirical counterparts of $\mathbb{T}(\cdot, \cdot)$ and the empirical marginals based on $\{(U_i, V_i)\}_{1 \leq i \leq n}$, respectively, by setting

$$\begin{aligned}
\mathbb{T}_n(u,v) &:= \frac{1}{n}\sum_{i=1}^{n} 1\!\mathrm{I}\{U_i \leq u, V_i \leq v\} := \mathbb{F}_n(G^{-1}(u), H^{-1}(v)), && (2.3)\\
\mathbb{U}_n(u) &= \mathbb{T}_n(u,1) = G_n(G^{-1}(u)), && (2.4)\\
\mathbb{V}_n(v) &= \mathbb{T}_n(1,v) = H_n(H^{-1}(v)). && (2.5)
\end{aligned}$$

The empirical quantile functions of $\mathbb{U}_n(\cdot)$ and $\mathbb{V}_n(\cdot)$ are given, for $n \geq 1$ and $0 \leq u, v \leq 1$, by

$$\begin{aligned}
\mathbb{U}_n^{-1}(u) &= \inf\{s \geq 0 : \mathbb{U}_n(s) \geq u\} = G(G_n^{-1}(u)), && (2.6)\\
\mathbb{V}_n^{-1}(v) &= \inf\{t \geq 0 : \mathbb{V}_n(t) \geq v\} = H(H_n^{-1}(v)). && (2.7)
\end{aligned}$$

Consider the empirical processes defined, respectively, for each $n \geq 1$ and $0 \leq u, v \leq 1$, by

$$\begin{aligned}
\alpha_n(u,v) &= n^{1/2}(\mathbb{T}_n(u,v) - uv), && (2.8)\\
\alpha_{n;\mathbb{U}}(u) &= \alpha_n(u,1) = n^{1/2}(\mathbb{U}_n(u) - u), && (2.9)\\
\alpha_{n;\mathbb{V}}(v) &= \alpha_n(1,v) = n^{1/2}(\mathbb{V}_n(v) - v), && (2.10)\\
\beta_{n;\mathbb{U}}(u) &= n^{1/2}(\mathbb{U}_n^{-1}(u) - u), && (2.11)\\
\beta_{n;\mathbb{V}}(v) &= n^{1/2}(\mathbb{V}_n^{-1}(v) - v). && (2.12)
\end{aligned}$$

In view of the definition (1.2) of $\mathbb{C}_n(\cdot,\cdot)$, the relation between $\mathbb{C}_n(\cdot,\cdot)$ and $\mathbb{T}_n(\cdot,\cdot)$ is given, for each $n \geq 1$ and $0 \leq u, v \leq 1$, by

$$\mathbb{T}_n(\mathbb{U}_n^{-1}(u), \mathbb{V}_n^{-1}(v)) = \mathbb{F}_n(G_n^{-1}(u), H_n^{-1}(v)) = \mathbb{C}_n(u,v). \tag{2.13}$$

In order to study the process $\mathbb{G}_n^*(\cdot,\cdot)$ defined in (1.4), and in view of (2.9)-(2.12), we define the tail empirical and quantile processes in terms of the sequence $\{k_n\}_{n=1}^{\infty}$, for each $n \geq 1$ and $0 \leq u, v \leq 1$, by

$$\alpha_n^*(u,v) := \alpha_n\left(u\frac{k_n}{n}, v\frac{k_n}{n}\right), \tag{2.14}$$

$$\alpha_{n;\mathbb{U}}^*(u) := \left(\frac{k_n}{n}\right)^{-1/2} \alpha_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right), \tag{2.15}$$

$$\alpha_{n;\mathbb{V}}^*(v) := \left(\frac{k_n}{n}\right)^{-1/2} \alpha_{n;\mathbb{V}}\left(\frac{vk_n}{n}\right), \tag{2.16}$$

$$\beta_{n;\mathbb{U}}^*(u) := \left(\frac{k_n}{n}\right)^{-1/2} \beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right), \tag{2.17}$$

$$\beta_{n;\mathbb{V}}^*(v) := \left(\frac{k_n}{n}\right)^{-1/2} \beta_{n;\mathbb{V}}\left(\frac{vk_n}{n}\right). \tag{2.18}$$

The tail of the empirical process of $\alpha_{n;\bullet}(\cdot)$ and the tail of the quantile process $\beta_{n;\bullet}(\cdot)$ play major and important role in statistics, for instance, the nonparametric statistics. This importance explains the huge variety of existing results in this fields, we may refer to Deheuvels (1997), Einmahl and Mason (1988a,b) and the references therein. The strong approximations of the processes $\alpha_n^*(\cdot,\cdot)$, $\alpha_{n;\bullet}^*(\cdot)$ and $\beta_{n;\bullet}^*(\cdot)$ are given in Mason and van Zwet (1987), Mason (1988), Csörgő and Horváth (1988).
Keep in mind the definition of $\mathbb{G}_n^*(\cdot,\cdot)$ in (1.4), for each $n \geq 1$ and $0 \leq u, v \leq 1$, we have

$$\begin{aligned}\mathbb{G}_n^*(u,v) &= \mathbb{G}_n\left(\frac{uk_n}{n}, \frac{vk_n}{n}\right)\\ &= n^{1/2}\left\{\mathbb{C}_n\left(\frac{uk_n}{n}, \frac{vk_n}{n}\right) - \frac{uk_n}{n}\frac{vk_n}{n}\right\}.\end{aligned}$$

This process can be decomposed as follows

$$\begin{aligned}&\mathbb{G}_n^*(u,v)\\ &= n^{1/2}\left\{\mathbb{T}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) - \frac{uk_n}{n}\frac{vk_n}{n}\right\}\\ &= \alpha_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) + n^{1/2}\left[\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right)\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right) - \frac{uk_n}{n}\frac{vk_n}{n}\right]\\ &= \alpha_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) + u\frac{k_n}{n}\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right) + v\frac{k_n}{n}\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\\ &\quad + n^{-1/2}\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right).\end{aligned} \tag{2.19}$$

Our approximations will be based on the following fact, which combines results of Einmahl and Mason (1988a) and Einmahl and Mason (1988b). For convenience, we will denote *sup-norm* of a bounded

function $f(\cdot)$, defined on $I = [0,1]$ or $I = [0,1]^2$, by $\|f\| = \sup_{x\in I}|f(x)|$. The next fact, due to Einmahl and Mason (1988a) provides a law of the iterated logarithm of the local quantile process.

Fact 1. Under (H.1)-(H.4), we have, with probability 1,

$$\begin{aligned}\limsup_{n\to\infty}(\log_2 n)^{-1/2}\|\beta^*_{n;\bullet}\| &= 2^{1/2}(1-\gamma)^{1/2} \quad \text{if } 0\le\gamma\le 1/2,\\ &= 2^{-1/2}\gamma^{-1/2} \qquad \text{if } 1/2<\gamma\le 1.\end{aligned} \tag{2.20}$$

Einmahl and Mason (1988b) established Bahadur-Kiefer type representation, which relates to the sum of the local uniform empirical process and the local quantile process. Let

$$\mathbf{R}_{n;\bullet}(k_n) := \sup_{s\in[0,\frac{k_n}{n}]}|\alpha_{n;\bullet}(s)+\beta_{n;\bullet}(s)|, \tag{2.21}$$

$$r_n := n^{-1/2}k_n^{1/4}(\log_2 n)^{1/4}\left(\log_1(k_n)+2\log_2 n\right)^{1/2}. \tag{2.22}$$

In the sequel, we need the following fact due to Einmahl and Mason (1988b).

Fact 2. Let $\{k_n\}_{n=1}^{\infty}$ a sequence of positive constants which satisfy the assumptions (H.1)-(H.4).

(i) When $\gamma = 0$, we have, with probability 1,

$$\limsup_{n\to\infty} r_n^{-1}\mathbf{R}_{n;\bullet}(k_n)\le 2^{1/4}. \tag{2.23}$$

In addition, when $\log_1(k_n)/\log_2 n\to\infty$ as $n\to\infty$, we obtain an equality on (2.23).

(ii) When $0<\gamma\le 1$, we have, with probability 1,

$$\begin{aligned}\limsup_{n\to\infty} r_n^{-1}\mathbf{R}_{n;\bullet}(k_n) &= 2^{1/4}(1-\gamma)^{1/4}, \quad 0<\gamma\le 1/2,\\ &= 2^{-1/4}\gamma^{-1/4}, \qquad 1/2<\gamma\le 1.\end{aligned}$$

In view of (2.6), (2.7), (2.17) and (2.18), we have

$$\mathbb{U}_n^{-1}(uk_n/n) := uk_n/n + n^{-1}k_n^{1/2}\beta^*_{n;\mathbb{U}}(u), \tag{2.24}$$

$$\mathbb{V}_n^{-1}(vk_n/n) := vk_n/n + n^{-1}k_n^{1/2}\beta^*_{n;\mathbb{V}}(v). \tag{2.25}$$

Consider now the modulus of continuity $w_n(\cdot)$ of $\alpha_n(\cdot,\cdot)$ defined by

$$w_n(h_n) := \sup_{L\in\mathcal{R}:|L|\le h_n} |\,\alpha_n(L)\,| \quad \text{for } h_n\in(0,1), \tag{2.26}$$

where

$$\begin{aligned}\mathcal{R} &:= \{[\mathbf{s},\mathbf{t}] = [s_1,t_1]\times[s_2,t_2] : 0\le s_i\le t_i\le 1 \text{ for } i=1,2\},\\ |L| &= |\mathbf{t}-\mathbf{s}| = \prod_{i=1}^{2}|t_i-s_i|\end{aligned}$$

and h_n denotes a sequence of positive constants fulfilling the conditions of the following fact due to Einmahl and Ruymgaart (1987).

Fact 3. Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of positive numbers on $(0,1)$ with $h_n \downarrow 0$ as $n \to \infty$, such that

$$i) nh_n \uparrow \infty, \quad ii) nh_n/\log_1 n \to \infty, \quad iii) \log_1(1/h_n)/\log_2 n \to \infty.$$

Then, with probability 1,

$$\lim_{n\to\infty} (2h_n \log_1(1/h_n))^{-1/2} w_n(h_n) = 1. \tag{2.27}$$

The proof of our result relies on the following oscillation inequality for bivariate empirical process, which is mentioned in Deheuvels *et al.* (2006).

Fact 4. For $0 \leq u_1, v_1, u_2, v_2 \leq 1$, we have

$$\mid \alpha_n(u_1, v_1) - \alpha_n(u_2, v_2) \mid \leq 3 \times w_n(|u_1 - u_2| \vee |v_1 - v_2|). \tag{2.28}$$

For each $n \geq 1$ and $0 \leq u, v \leq 1$, set

$$\begin{aligned} \alpha^*_{n;0}(u,v) &:= \alpha_{n;0}\left(\frac{uk_n}{n}, \frac{vk_n}{n}\right) \\ &:= \alpha_n\left(\frac{uk_n}{n}, \frac{vk_n}{n}\right) - u\frac{k_n}{n}\alpha_{n,\mathbb{V}}\left(v\frac{k_n}{n}\right) - v\frac{k_n}{n}\alpha_{n,\mathbb{U}}\left(u\frac{k_n}{n}\right). \end{aligned} \tag{2.29}$$

We are now in position to study the behavior of the empirical copula process on $[0, \frac{k_n}{n}] \times [0, \frac{k_n}{n}]$. Our main result is summarized in the following theorem.

Theorem 2.1 *Let* $\{k_n\}_{n\geq 1}$ *be a sequence of positive numbers fulfilling the assumptions* (H.1)-(H.4). *We have almost surely,*

(i) when $0 < \gamma \leq 1/2$*,*

$$\begin{aligned} &\limsup_{n\to\infty} n^{1/2} k_n^{-1/4} (\log_2 n)^{-1/4} (\log_1 n)^{-1/2} \|\mathbb{G}^*_n - \alpha^*_{n;0}\| \\ &\qquad \leq [3 \times 2^{-1/4} + \gamma 2^{5/4}](1-\gamma)^{1/4}, \end{aligned} \tag{2.30}$$

(ii) when $1/2 < \gamma \leq 1$*,*

$$\begin{aligned} &\limsup_{n\to\infty} n^{1/2} k_n^{-1/4} (\log_2 n)^{-1/4} (\log_1 n)^{-1/2} \|\mathbb{G}^*_n - \alpha^*_{n;0}\| \\ &\qquad \leq [3 \times 2^{-3/4} + \gamma 2^{3/4}]\gamma^{-1/4}. \end{aligned} \tag{2.31}$$

Proof.

For each $n \geq 1$ and $0 \leq u, v \leq 1$, we have

$$\begin{aligned}
\mathbb{G}_n^*(u,v) - \alpha_{n;0}^*(u,v) &= \left[\alpha_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) - \alpha_n\left(u\frac{k_n}{n}, v\frac{k_n}{n}\right)\right] \\
&\quad + v\frac{k_n}{n}\left[\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right) + \alpha_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\right] \\
&\quad + u\frac{k_n}{n}\left[\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right) + \alpha_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right)\right] \\
&\quad + n^{-1/2}\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right) \\
&= R_{n;0}(u,v) + v\frac{k_n}{n}R_{n;\mathbb{U}}(u) + u\frac{k_n}{n}R_{n;\mathbb{V}}(v) + R_n(u,v).
\end{aligned}$$

We study each quantity separately. For the particular choice of $u_1 = \mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right)$, $u_2 = u\frac{k_n}{n}$, $v_1 = \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)$ and $v_2 = v\frac{k_n}{n}$ in (2.28), combined with (2.24) and (2.25), we get that, with probability 1, for n sufficiently large,

$$\|R_{n;0}\| \leq 3 \times w_n(|u_1 - u_2| \vee |v_1 - v_2|).$$

Observe that

$$|u_1 - u_2| = n^{-1}k_n^{1/2}\|\beta_{n;\mathbb{U}}^*\| \quad \text{et} \quad |v_1 - v_2| = n^{-1}k_n^{1/2}\|\beta_{n;\mathbb{V}}^*\|.$$

First, we consider the case when $0 \leq \gamma \leq 1/2$. From the Fact 1, we infer that, almost surely, for n sufficiently large,

$$\|\beta_{n;\mathbb{U}}^*\| = \|\beta_{n;\mathbb{V}}^*\| = \mathcal{O}\left((\log_2 n)^{1/2}\right),$$

hence,

$$|u_1 - u_2| \vee |v_1 - v_2| = \mathcal{O}\left(n^{-1}k_n^{1/2}(\log_2 n)^{1/2}\right).$$

Fix any $\epsilon > 0$, and set

$$h_n = (1+\epsilon)n^{-1}[2(1-\gamma)]^{1/2}k_n^{1/2}(\log_2 n)^{1/2}.$$

By (2.27), we have almost surely,

$$\begin{aligned}
\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}w_n(|u_1 - u_2| \vee |v_1 - v_2|) \\
= 2^{-1/4}(1-\gamma)^{1/4}\sqrt{1+\epsilon}.
\end{aligned}$$

Consequently, we have almost surely,

$$\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|R_{n;0}\| \leq 3 \times \left[\frac{1-\gamma}{2}\right]^{1/4}. \tag{2.32}$$

Thought the sequel, set

$$\mathbf{V}_n := n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}.$$

To study the second term $R_{n;\mathbb{U}}(\cdot)$, let us recall from (2.21), (2.22) the definitions of $R_{n;.}(\cdot)$ and r_n. By Fact 2, we have, almost surely,

$$\begin{aligned}
&\limsup_{n\to\infty} \{\mathbf{V}_n \|R_{n;\mathbb{U}}\|\} \\
&\leq \limsup_{n\to\infty} \frac{k_n}{n} \times \left[\frac{\log_1(k_n) + 2\log_2 n}{\log_1 n}\right]^{1/2} r_n^{-1}\mathbf{R}_{n;\mathbb{U}}(k_n).
\end{aligned}$$

Under (H.1)-(H.4), and keeping in mind that the condition (H.3) implies that $\left[\frac{\log_1(k_n)+2\log_2 n}{\log_1 n}\right]$ converges to 1 as $n \to \infty$, one can see that, almost surely,

$$\limsup_{n\to\infty} \{\mathbf{V}_n \|R_{n;\mathbb{U}}\|\} \leq 2^{1/4}\gamma(1-\gamma)^{1/4}. \tag{2.33}$$

Similarly, using the same preceding arguments, we obtain, almost surely,

$$\limsup_{n\to\infty} \{\mathbf{V}_n \|R_{n;\mathbb{V}}\|\} \leq 2^{1/4}\gamma(1-\gamma)^{1/4}. \tag{2.34}$$

Since

$$\|\beta_{n;\mathbb{U}}\| = \|\beta_{n;\mathbb{V}}\| = \mathcal{O}\left((k_n \log_2 n)^{1/2}\right),$$

we then conclude that, almost surely,

$$\limsup_{n\to\infty} \{\mathbf{V}_n \|R_n\|\} = 0. \tag{2.35}$$

By combining (2.32), (2.33), (2.34) with (2.35) we obtain (2.45).
In the case when $1/2 < \gamma \leq 1$, by using the same arguments to proof (2.45) and choosing

$$h_n = (1+\epsilon)n^{-1}2^{-1/2}\gamma^{-1/2}k_n^{1/2}(\log_2 n)^{1/2}\log_1 n,$$

to obain (2.46). The proof of the Theorem is now completed. $\square$

Recall the following result due to Castelle and Laurent-Bonvalot (1998).

Theorem 2.2 *On a suitable probability space, one can construct a sequence* $\{\mathbf{B}_n(u,v) : (u,v) \in [0,1]^2\}$ *of copula Brownian Bridges such that we may define the bivariate empirical process* $\{\alpha_n(u,v) : (u,v) \in [0,1]^2\}$*, such that, for all positive* x *and all* a*,* $b \in [0,1]$*, the following inequality holds*

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq u\leq a, 0\leq v\leq b} |\alpha_n(u,v) - \mathbf{B}_n(u,v)| \geq n^{-1/2}(x + \Lambda_1 \log(nab))\log(nab)\right) \\
&\leq \Lambda_2 \exp(-\Lambda_3 x),
\end{aligned}$$

where Λ_1*,* Λ_2 *and* Λ_3 *are absolute constants.*

We define a sequence of *tied-down Brownian bridges*, with the same law of $\{\mathbb{B}(s,t) : (s,t) \in [0,1]^2\}$, by setting, for $n = 1, 2, \ldots,$ and $(s,t) \in [0,1]^2$,

$$\mathbb{B}_n(s,t) = \mathbf{B}_n(s,t) - s\mathbf{B}_n(1,t) - t\mathbf{B}_n(s,1). \tag{2.36}$$

By combining Theorems 2.1 and 2.2, we obtain the following corollary.

Corollary 2.3 *Assume that the conditions of Theorem 2.1 and Theorem 2.2 hold. We have, almost surely,*

(i) when $0 < \gamma \leq 1/2$,

$$\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\mathbb{G}_n^* - \mathbb{B}_n\| \\ \leq [3\times 2^{-1/4} + \gamma 2^{5/4}](1-\gamma)^{1/4}, \tag{2.37}$$

(ii) when $1/2 < \gamma \leq 1$,

$$\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\mathbb{G}_n^* - \mathbb{B}_n\| \\ \leq [3\times 2^{-3/4} + \gamma 2^{3/4}]\gamma^{-1/4}. \tag{2.38}$$

2.3 General case

In this subsection we discuss the case when X and Y are dependent, i.e., for $(u, v) \in [0,1]^2$,

$$\mathbb{C}(u,v) \neq uv.$$

Assume that $\mathbb{C}(\cdot,\cdot)$ is twice continuously differentiable on $(0,1)^2$ and all the partial derivatives of second order are continuous on $[0,1]^2$. Then, By applying a Taylor series expansion, we have

$$\begin{aligned}
&\mathbb{G}_n^{**}(u,v)\\
&= n^{1/2}\left\{\mathbb{T}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right),\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) - \mathbb{C}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\right\}\\
&= \widetilde{\alpha}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right),\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) + n^{1/2}\left[\mathbb{C}\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right),\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) - \mathbb{C}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\right]\\
&= \widetilde{\alpha}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right),\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) + n^{1/2}\left\{\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right) - \frac{uk_n}{n}\right\}\frac{\partial\mathbb{C}}{\partial u}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\\
&\quad + n^{1/2}\left\{\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right) - \frac{vk_n}{n}\right\}\frac{\partial\mathbb{C}}{\partial v}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\\
&\quad + \frac{n^{1/2}}{2}\left\{\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right) - \frac{uk_n}{n}\right\}^2\frac{\partial^2\mathbb{C}}{\partial u^2}(u',v') + \frac{n^{1/2}}{2}\left\{\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right) - \frac{vk_n}{n}\right\}^2\frac{\partial^2\mathbb{C}}{\partial v^2}(u',v')\\
&\quad + n^{1/2}\left\{\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right) - \frac{uk_n}{n}\right\}\left\{\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right) - \frac{vk_n}{n}\right\}\frac{\partial^2\mathbb{C}}{\partial u\partial v}(u',v'),
\end{aligned}$$

where (u', v') is a point between $\left(\frac{uk_n}{n}, \frac{vk_n}{n}\right)$ and $\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right)$, and

$$\widetilde{\alpha}_n(u,v) = n^{1/2}(\mathbb{T}_n(u,v) - \mathbb{C}(u,v)).$$

By using the preceding steps and facts, one finds

$$\begin{aligned}\mathbb{G}_n^{**}(u,v) &= \widetilde{\alpha}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) \\ &+n^{1/2}\left\{\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right)-\frac{uk_n}{n}\right\}\frac{\partial \mathbb{C}}{\partial u}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right) \\ &+n^{1/2}\left\{\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)-\frac{vk_n}{n}\right\}\frac{\partial \mathbb{C}}{\partial v}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right) \\ &+\mathcal{O}(n^{-3/2}k_n(\log_2 n)).\end{aligned}$$

For each $n \geq 1$ and $0 \leq u, v \leq 1$, set

$$\begin{aligned}\alpha_{n;0}^{**}(u,v) &:= \widetilde{\alpha}_{n;0}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right) \\ &:= \widetilde{\alpha}_n\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)-\frac{\partial \mathbb{C}}{\partial v}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\alpha_{n,\mathbb{V}}\left(v\frac{k_n}{n}\right) \\ &\quad -\frac{\partial \mathbb{C}}{\partial u}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\alpha_{n,\mathbb{U}}\left(u\frac{k_n}{n}\right).\end{aligned}$$

For each $n \geq 1$ and $0 \leq u, v \leq 1$, we have

$$\begin{aligned}&\mathbb{G}_n^{**}(u,v)-\alpha_{n;0}^{**}(u,v) \\ &= \left[\widetilde{\alpha}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right)-\widetilde{\alpha}_n\left(u\frac{k_n}{n},v\frac{k_n}{n}\right)\right] \\ &\quad +\frac{\partial \mathbb{C}}{\partial u}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\left[\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)+\alpha_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\right] \\ &\quad +\frac{\partial \mathbb{C}}{\partial v}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)\left[\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right)+\alpha_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right)\right] \\ &\quad +\mathcal{O}(n^{-3/2}k_n(\log_2 n)) \\ &:= R_{n;0}^{*}(u,v)+\frac{\partial \mathbb{C}}{\partial u}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)R_{n;\mathbb{U}}(u) \\ &\quad +\frac{\partial \mathbb{C}}{\partial v}\left(\frac{uk_n}{n},\frac{vk_n}{n}\right)R_{n;\mathbb{V}}(v)+\mathcal{O}(n^{-3/2}k_n(\log_2 n)).\end{aligned}$$

Using the fact that the first-order partial derivatives of a copula are bounded (see for instance Nelsen (2006)), i.e.,

$$0 \leq \frac{\partial \mathbb{C}(\cdot,\cdot)}{\partial u} \leq 1 \;\text{ and }\; 0 \leq \frac{\partial \mathbb{C}(\cdot,\cdot)}{\partial v} \leq 1,$$

one can find the following

$$\begin{aligned}|\mathbb{G}_n^{**}(u,v)-\alpha_{n;0}^{**}(u,v)| &\leq |R_{n;0}^{*}(u,v)|+|R_{n;\mathbb{U}}(u)|+|R_{n;\mathbb{V}}(v)| \\ &\quad +\mathcal{O}(n^{-3/2}k_n(\log_2 n)).\end{aligned}$$

Note that $|R_{n;0}^{*}(u,v)|$ may be treated making use the properties of the oscillation of the multivariate empirical process (Stute, 1984, Theorem 1.7). By combining all this with the preceding proof, we obtain the following result for the general case of copulas.

Theorem 2.4 *Let* $\{k_n\}_{n\geq 1}$ *be a sequence of positive numbers fulfilling the assumptions* (H.1)-(H.4). *Assume that* $\mathbb{C}(\cdot,\cdot)$ *is twice continuously differentiable on* $(0,1)^2$ *and all the partial derivatives of second order are continuous on* $[0,1]^2$. *We have almost surely,*

(i) when $0 < \gamma \leq 1/2$,

$$\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\mathbb{G}_n^{**} - \alpha_{n;0}^{**}\| \leq [3\times 2^{-1/4} + \gamma 2^{5/4}](1-\gamma)^{1/4}, \tag{2.39}$$

(ii) when $1/2 < \gamma \leq 1$,

$$\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\mathbb{G}_n^{**} - \alpha_{n;0}^{**}\| \leq [3\times 2^{-3/4} + \gamma 2^{3/4}]\gamma^{-1/4}. \tag{2.40}$$

Remark 2.5 *In the general case, i.e.,* $\mathbb{C}(u,v) \neq uv$, *the almost sure approximation rate is given, in Borisov (1982), by*

$$\|\widetilde{\alpha}_n - \mathbf{B}_n^*\| = \mathcal{O}(n^{-1/6}\log n), \tag{2.41}$$

where $\{\mathbf{B}_n^*(u,v) : (u,v) \in [0,1]^2, n \geq 1\}$ *, is a sequence of Brownian bridge, with covariance function*

$$\mathbb{E}(\mathbf{B}_n^*(s_1,t_1)\mathbf{B}_n^*(s_2,t_2)) = \mathbb{C}(s_1\wedge s_2, t_1\wedge t_2) - \mathbb{C}(s_1,t_1)\mathbb{C}(s_2,t_2), \quad \textit{for} \ \ (s_1,s_2,t_1,t_2) \in [0,1]^4.$$

Note that the rate in (2.41) is not

$$o\left(\frac{(\log n)^{1/2}(\log\log n)^{1/4}}{n^{1/4}}\right),$$

then, for the empirical copula process, we have the following

$$\|\mathbb{G}_n - \mathbb{B}_n^*\| = \mathcal{O}(n^{-1/6}\log n),$$

where

$$\mathbb{B}_n^*(s,t) = \mathbf{B}_n^*(s,t) - \mathbf{B}_n^*(1,t)\frac{\partial\mathbb{C}}{\partial t}(s,t) - \mathbf{B}_n^*(s,1)\frac{\partial\mathbb{C}}{\partial s}(s,t), \quad \textit{for} \ \ (s,t)\in[0,1]^2.$$

The results of Theorem 2.1 and Corollary 2.3 are the best possible and are governed by the almost sure rate of Bahadur-Kiefer representation given in Fact 2.

2.4 Application to tests of tail independence

This section is largely inspired from Deheuvels *et al.* (2006) and changes have been made in order to adopt it to our case. Our main concern, is testing the null hypothesis

$$\mathscr{H}_0 \ : \ \mathbb{C}(u,v) = uv, \quad \text{for} \quad (u,v)\in[0,k_n/n]^2.$$

First, we introduce weighted bivariate tests of tail independence. Namely, for selected constants ν_1 and $\nu_2 \in \mathbb{R}$, we set

$$\Omega_{n,k_n,\nu_1,\nu_2} = n \int_0^{k_n/n} \int_0^{k_n/n} u^{2\nu_1} v^{2\nu_2} \left\{ \mathbb{C}_n(u,v) - uv \right\}^2 dudv. \tag{2.42}$$

Recall the definition (2.36) of $\mathbb{B}_n(\cdot,\cdot)$, for $n \geq 1$. Therefore, by the triangle inequality,

$$\|\mathbb{G}_n^{*2} - \mathbb{B}_n^2\| \leq \|\mathbb{G}_n^* - \mathbb{B}_n\| \times \{2\|\mathbb{G}_n^*\| + \|\mathbb{G}_n^* - \mathbb{B}_n\|\}.$$

Note that the following elementary observation holds

$$\int_0^1 \int_0^1 u^{2\nu_1} v^{2\nu_2} dudv < \infty, \tag{2.43}$$

when $\nu_1 > -1/2$ and $\nu_2 > -1/2$.

We recall the following result given in (Deheuvels *et al.*, 2006, Corollary 2.1). We have, with probability one,

$$\limsup_{n\to\infty} (2\log\log n)^{-1/2} \|\mathbb{G}_n^*\| = \frac{1}{4}. \tag{2.44}$$

By combining (2.43), (2.43) and (2.44), one finds the following.

Corollary 2.6 *We have, almost surely, for $\nu_1 > -1/2$ and $\nu_2 > -1/2$,*

(i) when $0 < \gamma \leq 1/2$,

$$\begin{aligned} &\limsup_{n\to\infty} n^{1/2} k_n^{-1/4} (\log_2 n)^{-3/4} (\log_1 n)^{-1/2} \left| \Omega_{n,k_n,\nu_1,\nu_2} - \int_0^{k_n/n} \int_0^{k_n/n} \mathbb{B}_n^2(u,v) dudv \right| \\ &\quad \leq [3 \times 2^{-9/4} + \gamma 2^{-3/4}](1-\gamma)^{1/4}, \end{aligned} \tag{2.45}$$

(ii) when $1/2 < \gamma \leq 1$,

$$\begin{aligned} &\limsup_{n\to\infty} n^{1/2} k_n^{-1/4} (\log_2 n)^{-3/4} (\log_1 n)^{-1/2} \left| \Omega_{n,k_n,\nu_1,\nu_2} - \int_0^{k_n/n} \int_0^{k_n/n} \mathbb{B}_n^2(u,v) dudv \right| \\ &\quad \leq [3 \times 2^{-11/4} + \gamma 2^{-5/4}]\gamma^{-1/4}. \end{aligned} \tag{2.46}$$

3 Strong approximation of smoothed local empirical copula process

The smoothed empirical distribution function $\widehat{\mathbb{F}}_n(\cdot,\cdot)$ is defined, for each $n \geq 1$ and $x, y \in \mathbb{R}$, by

$$\widehat{\mathbb{F}}_n(x,y) := \frac{1}{n} \sum_{i=1}^{d} K_n(x - X_i, y - Y_i).$$

Here $K_n(x,y) = K(a_n^{-1/2}x, a_n^{-1/2}y)$, and

$$K(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} k(u,v)dudv$$

for some bivariate kernel function $k : \mathbb{R}^2 \mapsto \mathbb{R}$, with $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x,y)dxdy = 1$, and a sequence of bandwidths $a_n \downarrow 0$ as $n \to \infty$. For notational convenience, we have chosen the same bandwidth sequence for each margin. This assumption can easily be dropped. For small enough bandwidths a_n, the empirical distribution function $\mathbb{F}_n(\cdot,\cdot)$ and the smoothed empirical distribution function $\widetilde{\mathbb{F}}_n(\cdot,\cdot)$ are almost indistinguishable, for more details refer to (Fermanian *et al.*, 2004, Lemma 7) and also to van der Vaart (1994). The continuity of the marginals $F(\cdot)$ and $G(\cdot)$ entails that we can replace them by uniform distributions. We then have, for each $n \geq 1$ and $0 \leq u, v \leq 1$,

$$\widehat{\mathbb{T}}_n(u,v) := \frac{1}{n}\sum_{i=1}^{d} K_n(u - U_i, v - V_i),$$

and the smoothed empirical copula function

$$\widehat{\mathbb{C}}_n(u,v) = \widehat{\mathbb{T}}_n(G_n^{-1}(u), H_n^{-1}(v)).$$

Consider the empirical process defined, for each $n \geq 1$ and $0 \leq u, v \leq 1$, by

$$\begin{aligned} \widehat{\alpha}_n(u,v) &= n^{1/2}(\widehat{\mathbb{T}}_n(u,v) - uv), & (3.1)\\ \widehat{\alpha}_n^*(u,v) &:= \widehat{\alpha}_n\left(u\frac{k_n}{n}, v\frac{k_n}{n}\right), & (3.2)\end{aligned}$$

and define the smoothed empirical copula process $\widehat{\mathbb{G}}_n(\cdot,\cdot)$ by setting

$$\widehat{\mathbb{G}}_n(u,v) := n^{1/2}(\widehat{\mathbb{C}}_n(u,v) - uv), \quad \text{for } (u,v) \in [0,1]^2. \tag{3.3}$$

The main aim of this section is to investigate the smoothed empirical copula process defined, in terms of a sequence $\{k_n\}_{n=1}^{\infty}$, for each $n \geq 1$, by

$$\widehat{\mathbb{G}}_n^*(u,v) := \widehat{\mathbb{G}}_n\left(u\frac{k_n}{n}, v\frac{k_n}{n}\right) \quad \text{for } 0 \leq u, v \leq 1. \tag{3.4}$$

Observe that

$$\begin{aligned} \widehat{\mathbb{G}}_n^*(u,v) &= n^{1/2}\left\{\widehat{\mathbb{T}}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) - \frac{uk_n}{n}\frac{vk_n}{n}\right\}\\ &= \widehat{\alpha}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right)\\ &\quad + n^{1/2}\left[\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right)\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right) - \frac{uk_n}{n}\frac{vk_n}{n}\right]\\ &= \widehat{\alpha}_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right), \mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right) + u\frac{k_n}{n}\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right)\\ &\quad + v\frac{k_n}{n}\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right) + n^{-1/2}\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right). \end{aligned} \tag{3.5}$$

We shall assume that
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)^{1/2}dK(x,y)<\infty. \tag{3.6}$$
Since $\mathbb{T}(\cdot,\cdot)$ satisfies condition (2.1) in Mason and Swanepoel (2010), one can apply (Mason and Swanepoel, 2010, Corollary 2, eq (2.7)) to obtain
$$\|\widehat{\mathbb{T}}_n-\mathbb{T}_n\|=O\left(\frac{\sqrt{a_n\log_2 n}}{\sqrt{n}}\right),\quad a.s., \tag{3.7}$$
which gives
$$\begin{aligned}\widehat{\mathbb{G}}_n^*(u,v) &= \alpha_n\left(\mathbb{U}_n^{-1}\left(\frac{uk_n}{n}\right),\mathbb{V}_n^{-1}\left(\frac{vk_n}{n}\right)\right)\\ &+u\frac{k_n}{n}\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right)+v\frac{k_n}{n}\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\\ &+n^{-1/2}\beta_{n;\mathbb{U}}\left(u\frac{k_n}{n}\right)\beta_{n;\mathbb{V}}\left(v\frac{k_n}{n}\right)\\ &+O\left(\sqrt{a_n\log_2 n}\right),\quad a.s.\end{aligned} \tag{3.8}$$
Here is our main result concerning $\widehat{\mathbb{G}}_n^*(\cdot,\cdot)$.

Theorem 3.1 *Let* $\{k_n\}_{n\geq 1}$ *be a sequence of positive numbers fulfilling the assumptions* (H.1)-(H.4) *and* $K(\cdot,\cdot)$ *satisfies (3.6). For* $a_n=O(n^{-(1/4+\delta)})$, $\delta>0$, *we have almost surely,*

(i) when $0<\gamma\leq 1/2$,
$$\begin{aligned}&\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\widehat{\mathbb{G}}_n^*-\alpha_{n;0}^*\|\\ &\qquad\leq[3\times 2^{-1/4}+\gamma 2^{5/4}](1-\gamma)^{1/4},\end{aligned} \tag{3.9}$$

(ii) when $1/2<\gamma\leq 1$,
$$\begin{aligned}&\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\widehat{\mathbb{G}}_n^*-\alpha_{n;0}^*\|\\ &\qquad\leq[3\times 2^{-3/4}+\gamma 2^{3/4}]\gamma^{-1/4}.\end{aligned} \tag{3.10}$$

We have immediately.

Corollary 3.2 *Under the conditions of preceding theorem and Theorem 2.2, we have almost surely,*

(i) when $0<\gamma\leq 1/2$,
$$\begin{aligned}&\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\widehat{\mathbb{G}}_n^*-\mathbb{B}_n\|\\ &\qquad\leq[3\times 2^{-1/4}+\gamma 2^{5/4}](1-\gamma)^{1/4},\end{aligned} \tag{3.11}$$

(ii) when $1/2<\gamma\leq 1$,
$$\begin{aligned}&\limsup_{n\to\infty} n^{1/2}k_n^{-1/4}(\log_2 n)^{-1/4}(\log_1 n)^{-1/2}\|\widehat{\mathbb{G}}_n^*-\mathbb{B}_n\|\\ &\qquad\leq[3\times 2^{-3/4}+\gamma 2^{3/4}]\gamma^{-1/4}.\end{aligned} \tag{3.12}$$

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